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LIEBER, Lillian R
Non-Euclidean geometry

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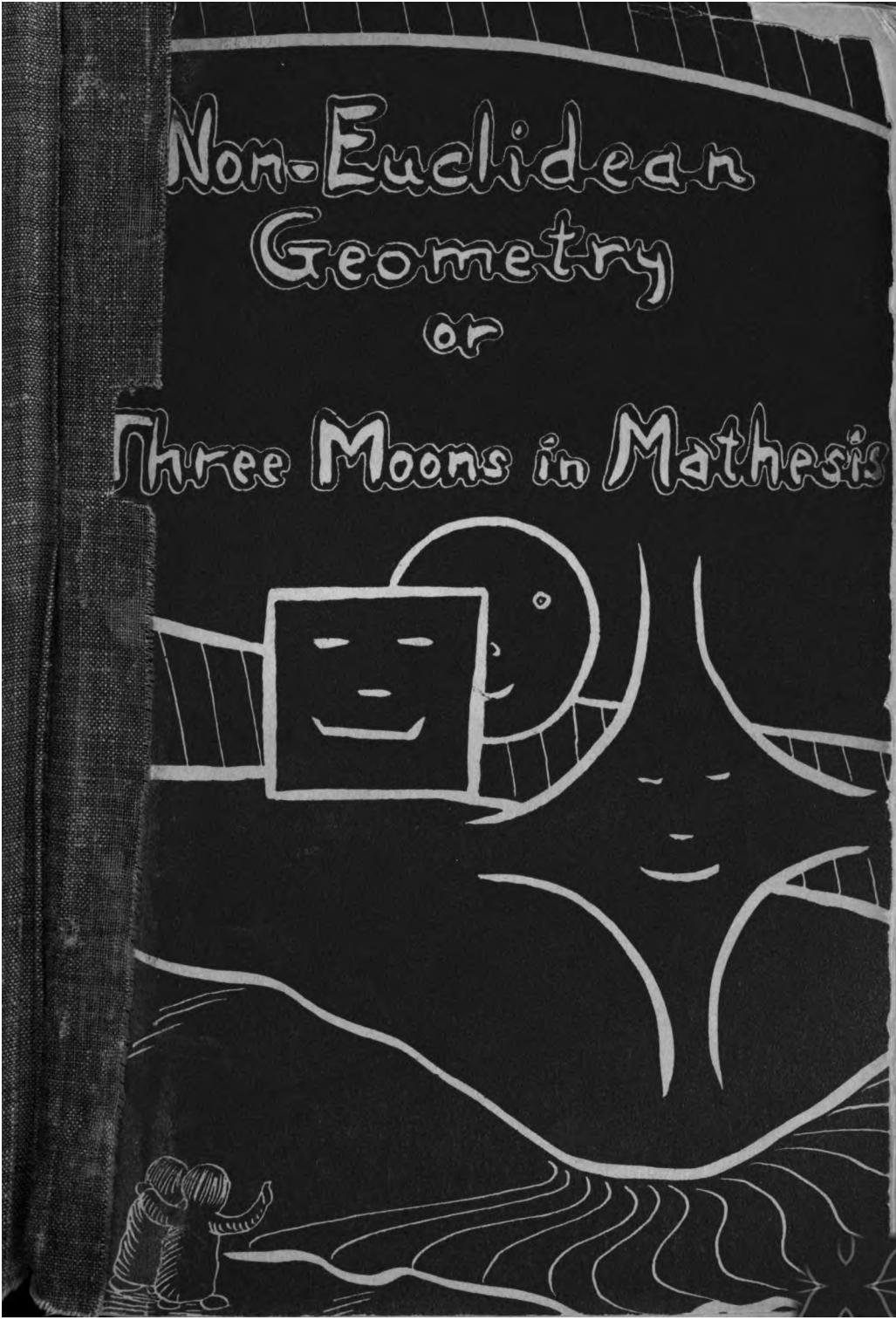
Non-Euclidean Geometry or Three Moons in Mathesis



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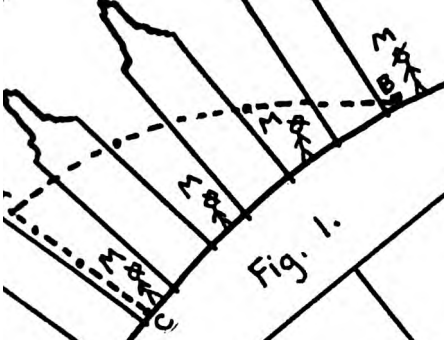


Fig. 1.

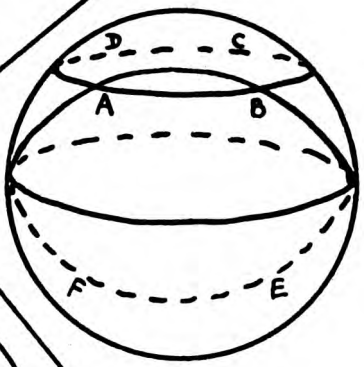


Fig. 5.

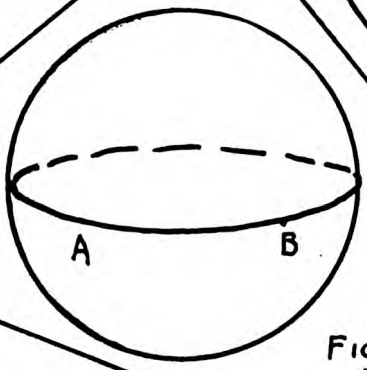


Fig. 2.

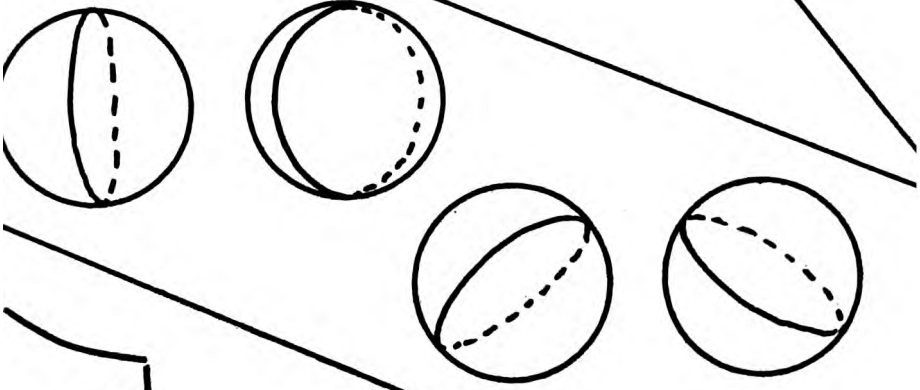
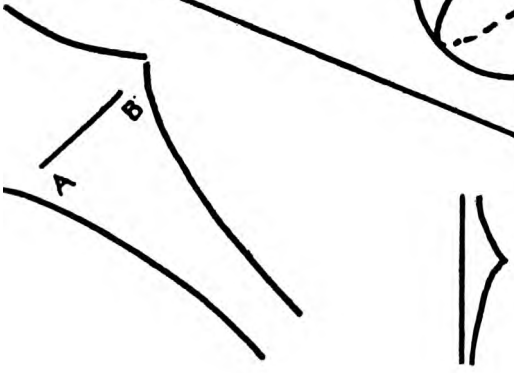


Fig. 3.



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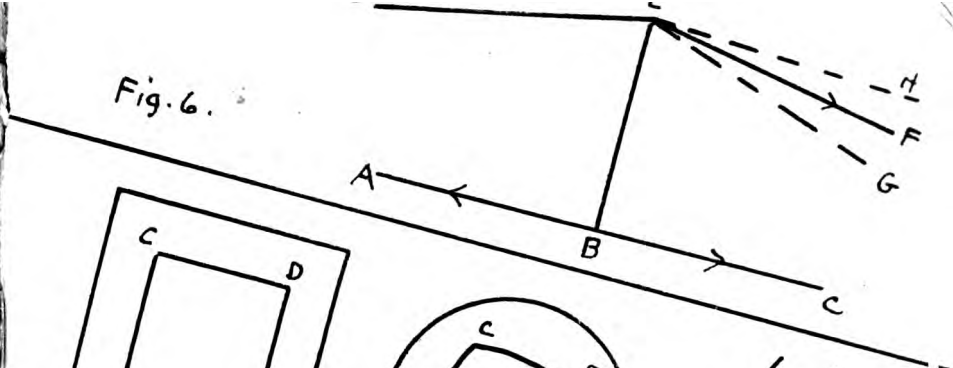
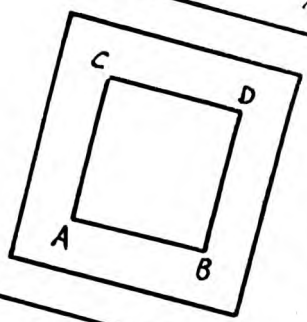
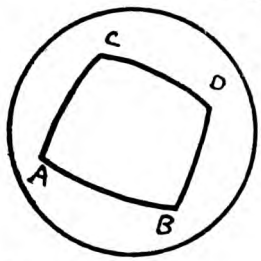






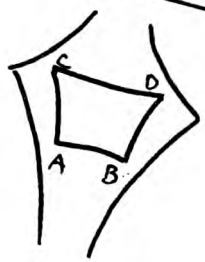
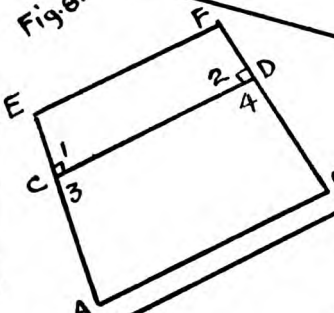


Fig. 7.

Fig. 8.



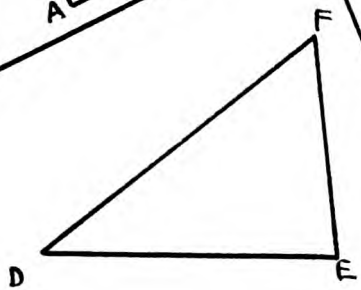
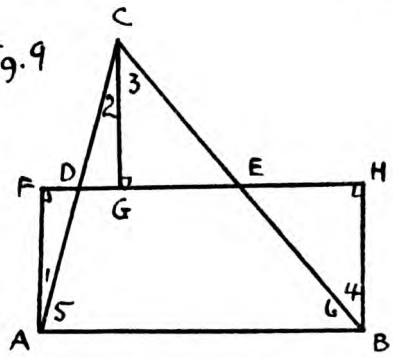


Fig. 9



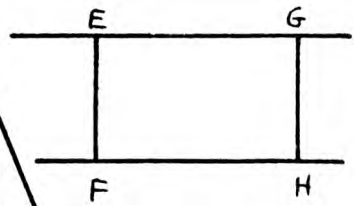


Fig. 11.

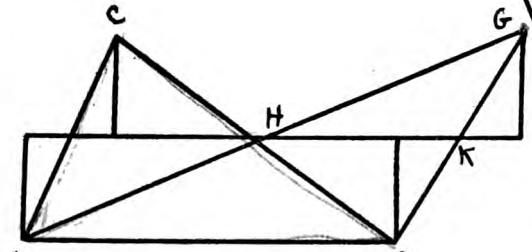


Fig. 10.

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Non-Euclidean Geometry
or
Three Moons in Mathesis

by
Lillian R. Lieber

with drawings
by
Hugh Gray Lieber

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Academy Press, New York.

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PREFACE

This is not intended to be
Free verse.
Writing each phrase on a separate line
Facilitates rapid reading,
And everyone
Is in a hurry
Nowadays.

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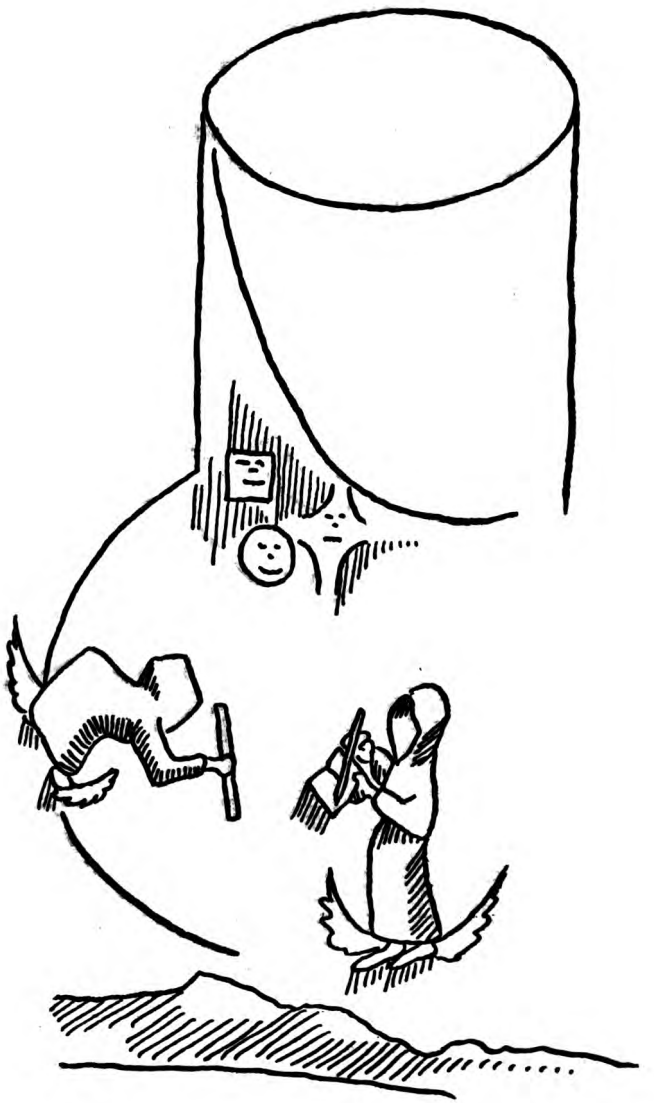

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NON-EUCLIDEAN GEOMETRY.

I. Introductory.

Plane geometry,
As everyone knows,
Is the study of certain figures
(Like the triangle, quadrilateral, circle, etc.)
Drawn on a plane.

This subject has been studied
For many centuries,
Particularly
Since it was put together
In systematic form
By Euclid
About 300 B.C.

From a few simple ideas or "postulates"
Like
"A straight line is the
Shortest distance
Between two points"
Euclid showed how to build up,
By means of logic,
Many complicated theorems,
As for example,
"In a right triangle,
The square of the hypotenuse
Is equal to
The sum of the squares of the legs",
As any high school boy now knows.

If anyone should ask
"How do you know
That a straight line is the
Shortest distance
Between two points?"

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Many people would undoubtedly say
That it is obvious;
But a little thought will show
That it is far from obvious.

For example,
Suppose something falls out of a window;
It certainly seems to fall in a straight line,
Does it not?
But, remembering
That the house itself is moving
As the earth, with everything on it,
(Including the falling object)
Turns on its axis,
The path of the falling object
Is the curve AB (See Fig. 1)
Rather than the straight line AC!
And if we should take into account also
The motion of the earth around the sun,
And the motion of the solar system,
(And are there not still other motions Unknown to us?)
We would see
That the path of the falling object
Is far from being the straight line
Seen by M, (Fig. 1)
The celebrated "Man-in-the-street."

Thus, what is a straight line
To one man,
Is not straight to another,
Depending upon the point of view.

"One man's meat
Is another man's
Poison."

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Consider now
The shortest distance between
Two points on a sphere,
A and B (Fig. 2).

Since it is impossible
To draw a straight line on a sphere,
The shortest distance on the surface
Must be a curve;
And as is well-known,
It is along the arc of a great circle
Through the given points.
(A great circle is formed
By cutting a sphere through its center—
Thus the equator and the meridians
Are great circles,
Though they are not the only ones possible,
Since a sphere can be cut
Through its center
In many other ways. See Fig. 3).

So, although we cannot be sure
Whether a path is a straight line or not,
We can at least agree
On the shortest distance
Between two points.
The nature of this shortest distance
Naturally depends upon the kind of surface
We are dealing with—
Thus, on a sphere,
The shortest distance is along an arc
Of a great circle;
On other surfaces,
As, for example, a pseudo-sphere*

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* See Glossary and Fig. 4.

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The shortest distance
Is another curve.

In general,
The shortest distance between two points
Is called a geodesic,
And its shape depends upon the surface
Upon which it is drawn.
Or, if you please,
You may call a geodesic
"A straight line"
Since this is relative anyway.

Another one of the postulates
Of Euclidean geometry
Is that
"Through a given point
Only one line can be drawn which is
Parallel to a given line."
This also did not seem "obvious"
To many mathematicians,
And, over a period of several centuries,
Attempts were made to prove it,
Without success.
It finally became clear
That this could not be proved
Because it did not follow
From the other postulates
And was therefore independent of them.

What would happen
If this particular postulate
Were contradicted
But all the remaining postulates retained?
This question was answered,
But not until the 19TH CENTURY!
It was found that

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Three entirely different geometries
Could be built up
Depending upon the assumptions that
(1) One and only one line
Can be drawn through a point
Parallel to a given line.
Or
(2) No line can be drawn
Through a given point
Parallel to a given line.
Or
(3) Two lines can be drawn
Through a given point
Parallel to a given line.

The geometries that arose from (2) and (3)
Were developed by
Riemann and Lobachevsky,* respectively,
And are called
Non-Euclidean geometries
Because the above-mentioned
Euclidean assumption (1)
Is contradicted in them.

But it should be clearly understood
That the other Euclidean postulates
Were retained
By both Riemann and Lobachevsky.

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* The geometry involving (3)
Was also developed, independently,
By Bolyai
And to some extent
By Gauss.

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Thus Non-Euclidean geometry
Has a technical meaning
In that it does not mean
Any geometry which is merely different from Euclid's,
But specifically,
A geometry which is like Euclid's in all respects
EXCEPT ONLY THAT
Assumption (1) is replaced
By either (2) or (3).

But there are many people
Who will say
That the idea of having no parallels
Or two parallels
Through the same point
Is unthinkable,
And they would rather
Accept the idea of a single parallel
Even if it cannot be proved.

But remembering that
The shortest distance between two points
On the surface of a sphere
Is an arc of a great circle
Let us then call such an arc
A "straight line"
For this surface.

We can now draw and study
Triangles, quadrilaterals, etc.,
And investigate the geometry
That we get
On a sphere.

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In the first place, it is obvious
That any two great circles intersect,
And consequently,
Since parallel lines can never meet,
IT IS IMPOSSIBLE TO HAVE
PARALLEL LINES ON A SPHERE.*

So, the idea of having no parallels
Is perfectly thinkable
Provided we visualize
The proper surface
And the proper kind of
"Straight lines" or
"Shortest distance lines".

Similarly,
On a pseudo-sphere (Fig. 4),
Through a given point
Many geodesics can be drawn
That never meet a given geodesic.

In particular,
If EF is one of these (Fig. 6)
Such that

---

* But are not the lines of latitude
Parallel lines?
No!
They are not "straight lines" at all,
Since the shortest distance
Between A and B (Fig. 5)
Is not along the latitude line ABCD,
But along the great circle ABEF,
And great circles
Always intersect each other.

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Any line within the angle FEB,
Such as EG,
Intersects BC,
And any line outside of this angle,
Such as EH,
Does not intersect BC,
Then EF is parallel to BC
In the direction indicated.
Likewise
ED is parallel to BA
In the other direction.
So that we have
TWO LINES THROUGH E,
Namely EF and ED,
BOTH PARALLEL TO THE SAME LINE
THROUGH THE SAME POINT E.

When EB is taken perpendicular to AC,
Then angle FEB,
Which is called the angle of parallelism,
Is less than 90 degrees.*
And even if EB
Is not perpendicular to AC,
The sum of angles FEB and EBC
IS LESS THAN 180° on a pseudo-sphere,
And NOT EQUAL to 180°, as is true on a plane!

And so the three assumptions
Given above,
Namely (1), (2), and (3) on page 6,
Apply on a PLANE, a SPHERE, and a PSEUDO-SPHERE, respectively.

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* How it is possible
To have this angle less than 90°
And still have EF and BC never meet
Will be seen more clearly on page 16.

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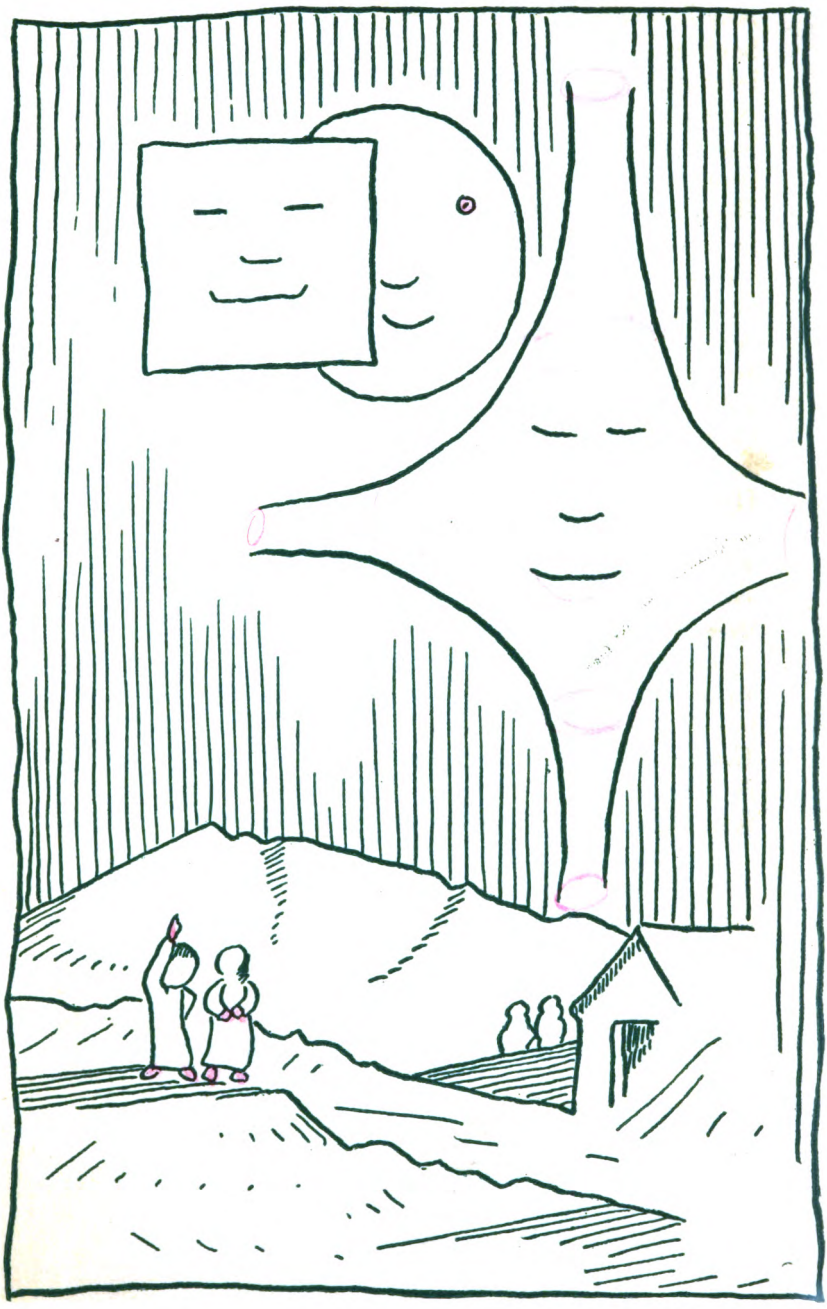

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II. Some Remarkable Theorems.

Let us now examine further
The three geometries
On these three surfaces.

A useful figure
That can be drawn
On any of the three surfaces
Is an
Isosceles birectangular quadrilateral.

In Fig. 7
Draw "a straight line," AB,
On each of the surfaces;
Erect equal perpendiculars at its ends;
Join C and D.
The reason for the name,
"Isosceles birectangular quadrilateral"
Is obvious.

Comparing the three figures,
We find that
(a) Angles C and D (the "summit angles")
    (1) are equal right angles on the plane,
    (2) equal obtuse angles on the sphere,
    (3) and equal acute angles on the pseudo-sphere.
(b) CD (the "summit") is
    (1) equal to AB, on the plane,
    (2) less than AB, on the sphere,
    (3) greater than AB on the pseudo-sphere.

It is easy to show
That the summit and summit angles
Are sufficient to determine
The quadrilateral
On a sphere or pseudo-sphere,

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But not on a plane,
For,
Suppose we could have
Two
Isosceles birectangular quadrilaterals,
EFDC and EFBA, (Fig. 8)
Then since angle 1 and angle 2
Are right angles by definition,
Their supplements,
Angle 3 and angle 4,
Are also right angles.
We should then have an
Isosceles quadrilateral ABDC
Having FOUR right angles,
Which is possible only on a plane.

Thus, on a sphere or pseudo-sphere,
If two isosceles birectangular quadrilaterals
Have the same summit and summit angles,
They are congruent.

By means of this quadrilateral
Several interesting theorems
About triangles
Can be derived, thus: (Fig. 9)
DE joins the mid-points
Of the sides AC and BC
Of triangle ABC.
AF, BH, and CG are perpendiculars
Dropped from the three vertices
Of the triangle ABC
To the line DE.

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By proving
Triangle AFD congruent to triangle CDG,
And
Triangle CGE congruent to triangle EHB,
We get

AF=CG=BH and hence AF=BH.

Therefore ABHF is an
Isosceles birectangular quadrilateral
Whose summit is AB
And summit angles 15 and 64.
From the above-mentioned triangles
We also have

angle 1 = angle 2
and angle 3 = angle 4.

Hence the sum of the three angles
Of triangle ABC
Equals the sum of the
Summit angles 15 and 64.

But the two summit angles
Add up to
(1) Two right angles or 180° on a plane,
(2) More than 180° on a sphere, and
(3) Less than 180° on a pseudo-sphere (page 10 (a))
Hence the interesting theorem that
THE SUM OF THE THREE ANGLES OF A TRIANGLE IS
(1) 180° on a plane,
(2) More than 180° on a sphere, and
(3) Less than 180° on a pseudo-sphere.

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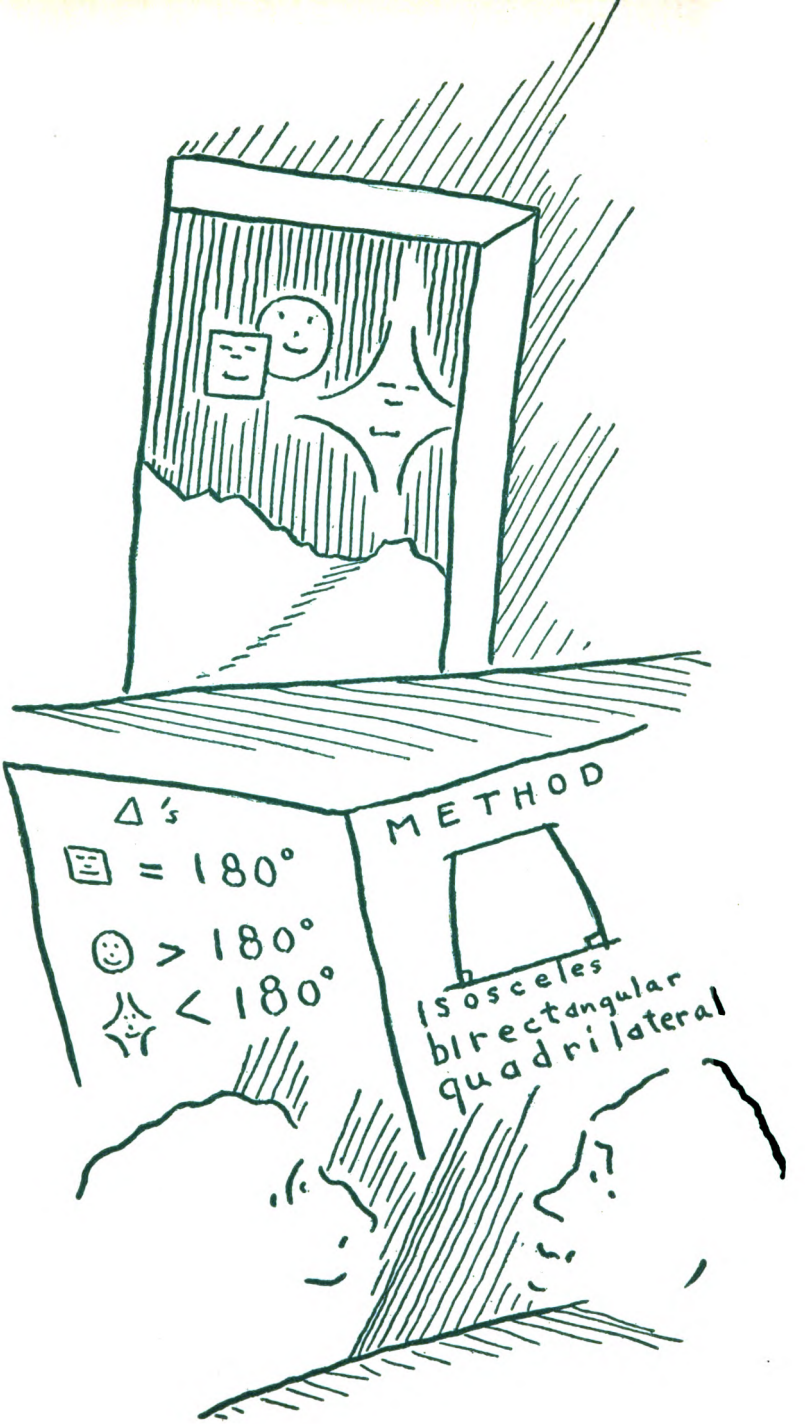

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We are so accustomed to the
Sum of the three angles of a triangle
Being equal to 180°, that
A contradiction of this seems preposterous,
And yet it is so easy to see that
Triangles drawn on
Various kinds of surfaces
Have a different angle sum!

Furthermore,
Since
Triangle AFD is congruent to triangle CDG
And
Triangle CGE is congruent to triangle EHB
It is clear that
The area of triangle ABC
Is equal to
The area of quadrilateral ABHF.

Hence the following
Simple but strange
Theorem:
On a sphere or a pseudo-sphere,
(But not on a plane)
Two triangles having
A side of one equal to a side of the other,
And having the same angle sum,
HAVE THE SAME AREA,
Since they are both equal to the same
Isosceles birectangular quadrilateral.*

---

* Since each is equivalent to
An isosceles birectangular quadrilateral
Having the same summit and summit angles,
And these are sufficient
To determine the quadrilateral (See p. 11).

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Indeed it can further be shown that
On these two surfaces
(Namely a sphere and a pseudo-sphere)
Two triangles having
Merely the same angle sum
Must have the same area.
Thus let triangles ABC and DEF (Fig. 10)
Have the same angle sum.
Construct triangle ABG making AG=DF, AH=HG
And having the same angle sum
As triangle ABC.*
Then triangles ABC and ABG have equal areas,
Because they have a side, AB, in common,
And the same angle sum (as shown above),
Also triangles ABG and DEF
Have equal areas,
Since they also have one side equal (AG=DF)
And the same angle sum.
Consequently,
Triangle ABC has the same area as triangle DEF.

Hence,
IF THREE ANGLES OF ONE TRIANGLE
ARE EQUAL RESPECTIVELY
TO THREE ANGLES OF ANOTHER
THE TRIANGLES HAVE EQUAL AREAS.†

---

* It is easy to prove K the mid-point of BG,
By means of congruent triangles.
Then ABC and ABG have
The same isosceles birectangular quadrilateral
And therefore the same angle sum.

† In other words
You cannot alter the size of a triangle on a sphere
Or on a pseudo-sphere
Without distorting it—
A fact not true on a plane.

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This theorem is NOT like the one
We are accustomed to on a plane,
Namely,
If three angles of one triangle
Are equal respectively
To three angles of another,
The triangles are similar
But not necessarily equal in area!

Thus, on a sphere or a pseudo-sphere,
Similar triangles have the same size!

It would appear from this
That there could be no trigonometry
On a sphere or a pseudo-sphere,
Since trigonometry is based
Upon the properties
Of similar triangles of different sizes.
But it will be shown later (See page 23).
That in spite of this fact
There IS a trigonometry
On these surfaces,
Developed from a different point of view,
But in perfect harmony
With the trigonometry on a plane
With which we are familiar.

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III. Other Consequences.

And so we have seen
That on a sphere there are no parallel lines,
On a plane only one line
Can be drawn through a given point
Parallel to a given line,
And on a pseudo-sphere
There are two such lines.

Furthermore
Two parallel lines on a plane
Never meet and are everywhere equally distant,
Thus, EF=GH (Fig. 11)
BUT
ON A PSEUDO-SPHERE
TWO PARALLEL LINES APPROACH EACH OTHER
ASYMPTOTICALLY,
That is,
They get nearer and nearer
To each other,
But never meet (See Fig. 12)
And it is easy to show
That this is the case,
That is
That CD must be less than AB:
At a point F, midway between A and C (Fig. 12)
Draw FE perpendicular to AC.
Now angle 1, the angle of parallelism*,
Is less than 90° (See page 9),

---

* The angle of parallelism at a given point
Varies with the distance
From the point to the line.
Thus, if CD and EG are both parallel to AB (Fig. 13)
And if angle A is a right angle,
Then the angle af parallelism at E,
Namely angle 1,
Is less than the angle of parallelism at C.

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And its supplement, angle 32,
Must therefore be more than 90°
And of course greater than angle 1.
Hence if you fold the figure FCDE
On EF as an axis,
The two right angles at F will fit,
And FC will coincide with FA,
Since F is the mid-point of AC.
And since angle 32 is greater than angle 1
ED will obviously take the direction EG,
So that DC will necessarily coincide
With a part of BA (namely GA)
And DC is therefore less than BA.

Suppose now that two lines, AB and CD,
Are parallel on a pseudo-sphere,
And let EF be perpendicular to AB (Fig. 14)
Then, as we now know,
Angle 1 is less than 90°.
Let us now take a line EG so that
Angle 12 is a right angle.
Then, since EG is not within
The angle of parallelism,
It cannot intersect AB (See p. 9 and Fig. 6).
And yet, it is not parallel to AB,
Since angle 12 equals 90°.
Hence,
ON A PSEUDO-SPHERE
IT IS POSSIBLE TO DRAW LINES THAT
DO NOT INTERSECT AND ARE NOT PARALLEL!
Namely, two lines that have
A common perpendicular.

That is, angle 2,
And the nearer you go to AB
The more the angle of parallelism approaches 90°,
Though it never actually becomes 90°.

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But even more remarkable than this
Is the fact that
The converse is also true!

That is,
IF TWO LINES NEITHER INTERSECT
NOR ARE PARALLEL,
THEY MUST HAVE A COMMON PERPENDICULAR
SOMEWHERE,
Even if angle 12 (Fig. 14)
Is not 90°.

The method of proving this
Is in itself very interesting.
There are two possible cases:
(a) EF and GH may be of equal length (Fig. 15)
In this case, choose K
Mid-way between H and F
And draw LK perpendicular to AB.
Then fold LKFE on LK as an axis.
Since angles 4 and 5 are both
Right angles, they will fit;
KF will fit on KH;
Angle 6 will fit on angle 7.

FE will fit HG, being equal by hypothesis,
Consequently, LE will have to coincide
With LG,
And angle 2 with angle 3.
Hence, since angles 2 and 3
Are both equal and supplementary,
They must be right angles.
Thus LK is a common perpendicular
Of the lines AB and CD.

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But the more interesting case is
(b) Where EF and GH are not equal.
Take EF less than GH, (Fig. 16)
Move DEFB to the left,
Until angle EFB coincides with
Angle 23;
Then DEFB becomes D′E′HB′,
For,
EF, being less than GH, by hypothesis,
It will become E′H,
And it is easy to show that
E′D′ must intersect CD!*

Now from the point of intersection, M,
Drop a perpendicular to AB,
And call it MO;
Now move D′E′HB′ back
To its original position DEFB.
Obviously every point on E′D′
Will take some position on CD,
And therefore M, being a point on E′D′,
Will also take some position on CD,
Such as P,
So that MO becomes PR.

---

* Since AB and CD
Neither intersect nor are parallel,
The angle DEF
Is greater than the angle of parallelism at E
(See p. 9 and Fig. 6.)
Hence draw FK parallel to CD.
Then HK′ is parallel to E′D′,
But HK′ is not parallel to FK
Since angle 3 = angle 4,
And angle 3 is therefore supplementary to angle 5.
Hence a line through H parallel to FK
Must lie within angle 3;
Let HL then be parallel to FK;

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Thus we have reduced this case to case (a),
Since now MO = PR,
And therefore the common perpendicular
Is situated at a point mid-way between
O and R.

In other words,
Two lines on a pseudo-sphere
Which neither intersect nor are parallel,
At first approach each other
Until they reach a certain distance apart,
And then diverge (Fig. 17);
The shortest distance, EF,
Being the common perpendicular.

We shall now show that
Two lines that are not parallel,
EVENTUALLY DIVERGE,
Whether they are intersecting lines or not.

Consider first the case of two intersecting lines (Fig. 18).

Obviously DE cannot equal FG,
For if it were,
There would be a common perpendicular
To AB and AC
At a point mid-way between G and E,
But this is impossible because

Then HL is parallel to CD
Since CD is parallel to FK.
Therefore HK′ will intersect CD (See p. 9)
And finally E′D′ (being parallel to HK′)
Will also intersect CD.

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Two lines that have a common perpendicular,
Do not intersect (See p. 17 and Fig. 14);
And DE cannot be less than FG because
If it were
We could obviously find a line HK
(By going near enough to the point of intersection A)
Which is less than DE.
Hence, there would be a line
Somewhere between HK and FG
Which would equal DE;
But this would again give us
A common perpendicular,
Which is impossible.

Therefore,
TWO INTERSECTING LINES ON A
PSEUDO-SPHERE
DIVERGE MORE AND MORE,
JUST AS THEY DO ON A PLANE.

And furthermore,
TWO LINES WHICH NEITHER INTERSECT NOR
ARE PARALLEL,
MAY APPROACH EACH OTHER FOR A WHILE,
THAT IS,
UNTIL THEY REACH THEIR COMMON
PERPENDICULAR,
BUT EVENTUALLY,
THEY ALSO DIVERGE MORE AND MORE,
FOR:
If AB and CD neither intersect nor are parallel (Fig 19)
It is possible to draw a line EF
Intersecting CD and parallel to AB.
Now since EF and AB approach each other
More and more (See p. 16)
And EF and ED diverge more and more,
As was shown above,
Obviously CD and AB must diverge more and more.

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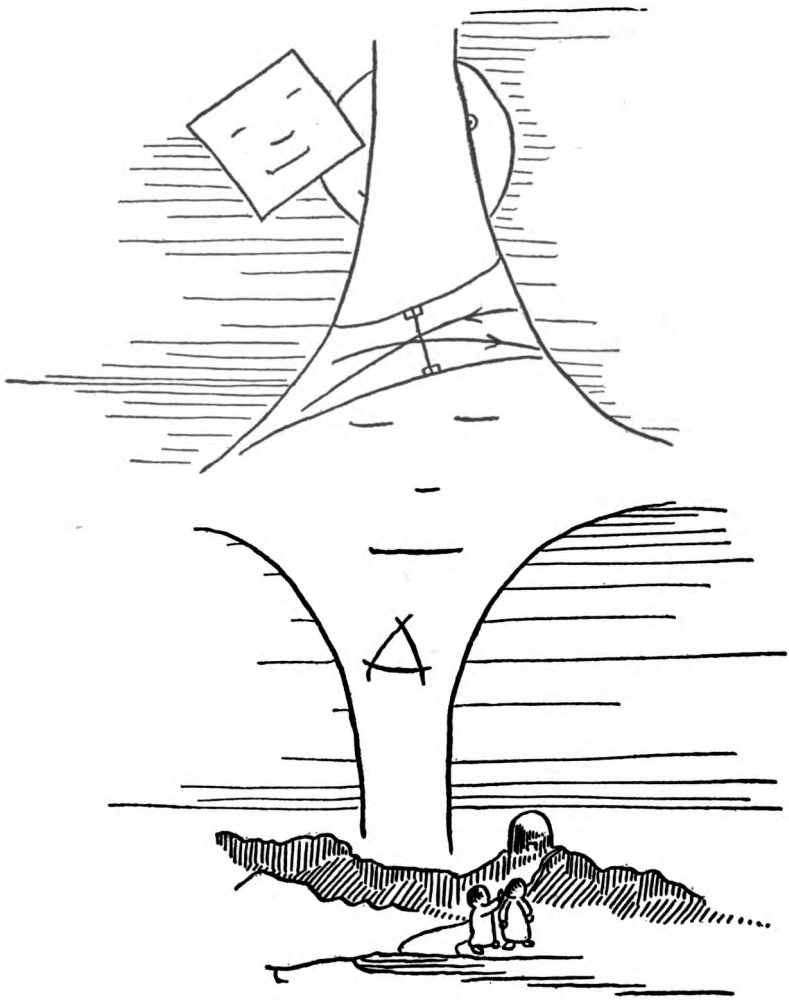

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Hence,
On a pseudo-sphere,
Two lines either approach each other
Asymptotically (See p. 16)
Or eventually diverge,
But are never everywhere equally distant
As are parallel lines on a Euclidean plane.

Some startling though perfectly intelligible facts
About lines on a sphere,
Are:

(a) All perpendiculars to the same straight line
Meet in a point at a constant distance,
Thus,
Take the equator as the given line;
Then the meridians are all
"Straight lines" perpendicular to the given line;
And they all meet at the poles!
This is of course very different
From the situation on a plane,
Where all perpendiculars to a given line
Are parallel to each other
And therefore NEVER MEET!

(b) All straight lines on a sphere
Have the same finite length!—
Namely the circumference of a great circle.
This is a very important point of difference
Between lines on a sphere
And those on a plane,
Since the latter (as well as those on a pseudo-sphere)
Are infinite in length.

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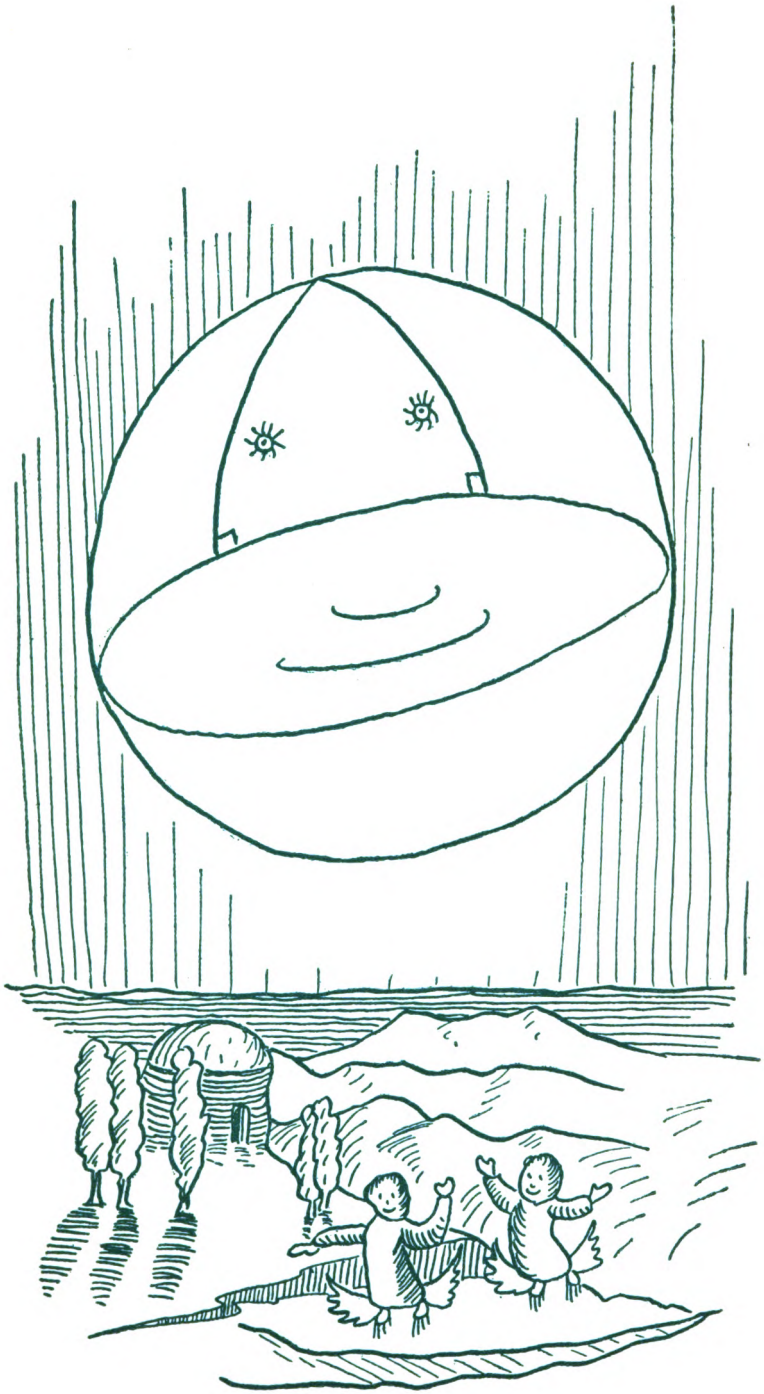

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IV. Non-Euclidean Trigonometry.

Having seen some of the theorems
Of Non-Euclidean geometry,
Theorems which may sound very strange
To someone familiar only with
Euclidean geometry,
But perfectly sensible to one
Who realizes that there are
Different geometries on different surfaces,
Such as a sphere or a psuedo-sphere,—
Let us now go back to the question
Mentioned on page 15
As to how it is possible
To have a trigonometry on these surfaces,
Although there are no similar triangles
Of different sizes.

The answer is that
Instead of defining AC/AB
As the cosine of angle A (Fig. 20)
As we do in plane trigonometry,
We now allow the triangle ABC to shrink
(Keeping a right angle at C),
And although now the ratio AC/AB is
No longer a constant as the triangle gets smaller,
Still it approaches a definite limit,
And it is this limit
Which is now called the cosine of angle A.

This is justified by the fact
That any function may be called a "cosine"
PROVIDED THAT
It satisfies the condition:

$$f(x+y) + f(x-y) = 2f(x)f(y),$$

(Which, by the way, is true of the
Ordinary cosine in plane trigonometry).

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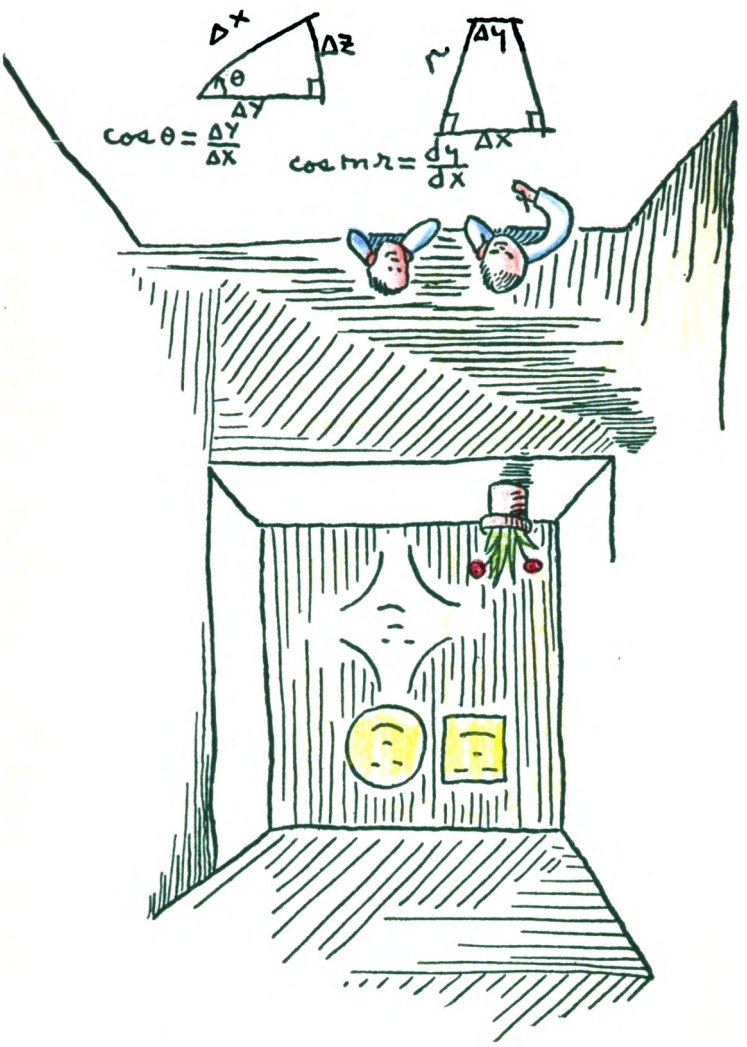
$$f(x+y) + f(x-y) = 2f(x) \cdot f(y)$$



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Furthermore,
Accepting this more general definition of cosine,
We find that if we have an
Isosceles birectangular quadrilateral,
ABCD (Fig. 21),
The limit of the ratio CD/AB
As AB approaches zero,
Keeping x constant,
Also satisfies the above condition
For a cosine function,
And therefore this limit
Is called the cosine of mx.

Thus in Non-Euclidean trigonometry
We have
Cosines of LINES as well as of ANGLES.*
And, as we shall see in a moment,
By giving proper values to m,
The cos mx can be made to apply
On a sphere or on a pseudo-sphere,
Or even on a plane!
For, as we now know,
(1) On a plane CD=AB
(2) On a sphere CD is less than AB
(3) On a speudo-sphere CD is greater than AB

[See p. 10 (b)]

Hence,
(1) On a plane CD/AB=1
(2) On a sphere CD/AB is less than 1
(3) On a pseudo-sphere CD/AB is greater than 1.

---

* A fact perfectly familiar
To anyone who has studied spherical trigonometry,
In which the sides of a triangle
As well as the angles,
Are given in degrees,
And quite naturally so,
Since lines on a sphere
Are really arcs of great circles.

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Consequently,

(1) On a plane, cos mx = 1, and therefore m = o, or m = 1/k, where k = infinity.

(2) On a sphere, cos mx is less than 1, and therefore m is real; hence, m=1/k where k is also real.

(3) On a pseudo-sphere, cos mx is greater than 1, which is possible only if m is imaginary!* Hence, m = i/k in this case, where i = $\sqrt{-1}$.

And now by quite simple means,†
It is possible to develop
A set of formulas
Which gives the relationship between
The sides and angles of a triangle.

---

* It will be remembered that
In plane trigonometry
The cosine of an angle cannot exceed 1.
But the hyperbolic cosine can be greater than 1,
The hyperbolic cosine of angle BOC (Fig. 22)
Being defined as OC/OA.
In other words,
The hyperbolic functions are defined
In connection with a hyperbola
Instead of with a circle,
As the ordinary trigonometric functions are,
The distance OA being used instead of a radius.
And furthermore,
The hyperbolic cosine of an angle x
Is equal to the ordinary cosine of ix.
Hence the m in (3) must be imaginary;
Thus the geometry on a pseudo-sphere
Is sometimes called
Hyperbolic geometry.

† See Monographs on Modern Mathematics, Edited by J. W. A. Young, p. 119.

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As, for example,

$$\cos mc = \cos ma \cos mb,$$

Where a, b, and c are the sides of a triangle,
And which apply equally well
ON A PLANE,
ON A SPHERE,
OR ON A PSEUDO-SPHERE,
By merely using the proper value of m!

Thus,
By taking $m = o$ for a plane,
We get the corresponding Euclidean formula
(Or merely the identity $1 = 1$ in some cases).

And so we have seen that
By using a more general definition of cosine
(Which includes the ordinary one as a special case),
It is possible
Not only to develop a trigonometry
On a sphere and pseudo-sphere,
But to do it in so general a way
That it includes
Ordinary trigonometry as a special case!

Just a word concerning analytics.
It is of course possible
To represent a point
On either a sphere or a pseudo-sphere
(As well as on a plane)
By a pair of coordinates.

Now,
These can be put in such a form
That they give quite simple equations
For the straight line, etc., etc.

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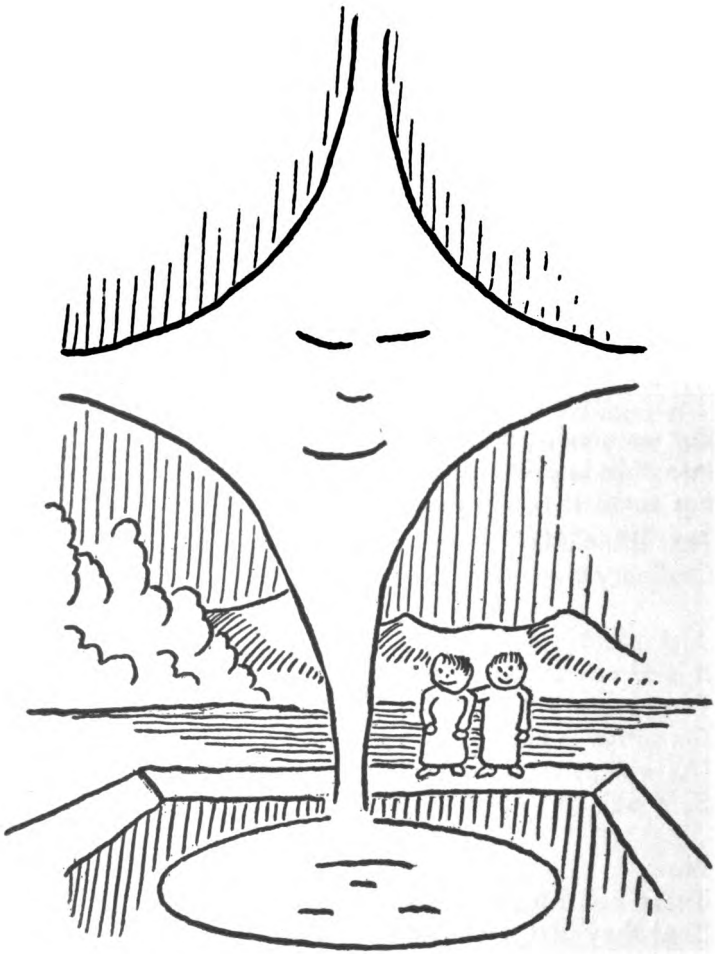

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One of the interesting relationships
Obtained here is that
If (x,y) represent any point
On a pseudo-sphere,
Then we get
The remarkable relationship:

$x^2 + y^2$ is less than k^2!

Hence if we plot these points
On a Euclidean plane,
They will all lie
Within a circle of radius k.
This means that
All the points of the entire pseudo-sphere
Are represented by
The points within a circle on a Euclidean plane.

But we cannot here go further
Into this fascinating part of the subject.
For further information
See the references given on page 31.

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V. A Postulate System.

It should be noted here that
Although the visualizing
Of the theorems
Of Non-Euclidean geometry and trigonometry
On a sphere or a pseudo-sphere
Makes it easier to understand them,
This visualization is really not necessary
(And sometimes even a hindrance)*
For the logical development of the subject.

In developing any subject logically,
All that is necessary is
To start with certain UNDEFINED† terms,
And certain UNPROVED theorems (or postulates)
And then by logical reasoning,
To derive from these
The entire structure of the subject.

Thus in geometry,
Whether Euclidean or Non-Euclidean,
The undefined terms may be
"Straight line," "plane," etc.,

---

* In dealing with Non-Euclidean Geometry
Of three or more dimensions,
Visualization is obviously impossible,
And the postulational method is the
Only approach.

† Obviously the terms at the BEGINNING of a subject
Are necessarily undefined,
Since there is nothing to define them by;
And similarly,
The foundation theorems or postulates
Cannot be proved
Since there is not yet any means
Whereby to prove them.

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Having certain properties
Described in the postulates—

As, for instance,
The postulate concerning
The number of lines that can be drawn
Through a given point parallel to a given line—
Without making any reference
To a sphere or a pseudo-sphere,
But merely calling the surface a "plane."

This is indeed the usual way
Of treating the subject,
But it is difficult for a beginner to realize
That "plane" IS an UNDEFINED term,
And he therefore always
Visualizes a Euclidean plane
Whenever the word "plane" is used;
And, of course, he is necessarily confused
Since he cannot see how it is possible
To have two lines parallel to a given line
Through the same point, etc., etc.

Thus if we develop a geometry
From the Riemannian postulates (See p. 6)
We get a set of theorems
Which we happen to know hold true on a sphere.
But have not these theorems
Been developed from EUCLIDEAN postulates
In ordinary spherical geometry?
The answer is that
By means of ordinary straight lines
And Euclidean planes in three dimensions,
We arrive at the same facts
Concerning the figures drawn on a sphere,
As we can do by
Riemannian "straight lines" and "planes"
In TWO dimensions.

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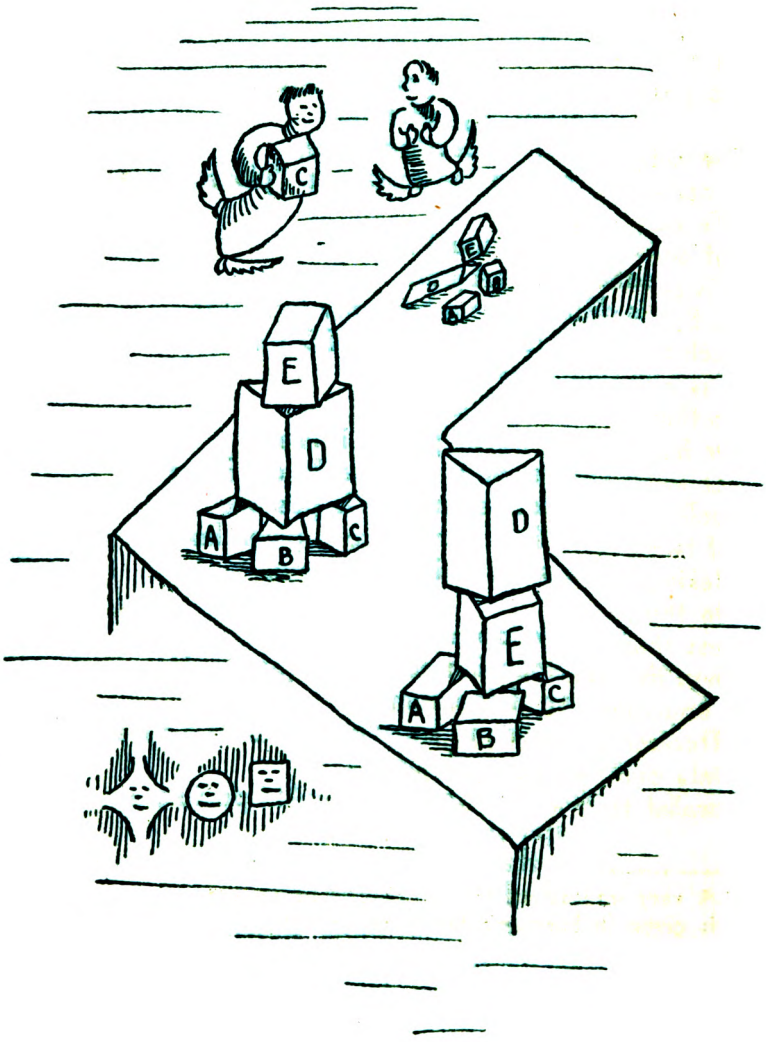

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That is, if we stay on the surface of a sphere,
Where we have no ordinary straight lines and
Euclidean planes,
We must use
The postulates appropriate for that surface—
As for example, that there are
No parallel "straight lines" on a sphere.

The subject of a postulate system
Is obviously very interesting and fundamental.*
We cannot go into it here,
But one important fact should be mentioned:
It is possible to have a set of postulates
A, B, C, and D,
Such that
D is equivalent to some other postulate, E,
So that from A, B, C, and D we can prove E,
Or from A, B, C, and E, we can prove D.
For instance,
Euclid's postulate:
"If two lines are cut by a transversal
Making the sum of the interior angles
On the same side of the transversal
Less than 180°,
Then the two lines will meet on that side,"
Is equivalent to saying that
"Through a given point
Only one line can be drawn
Parallel to a given line."

---

* A very interesting treatment of this subject
Is given in Fundamental Concepts of
Algebra and Geometry, by J. W. Young.

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But here is the important point:
These two statements are equivalent
Only PROVIDED that
The remaining Euclidean postulates
Are retained.

In other words,
We cannot say that
One statement is equivalent to another
Without also describing the environment
In which this is true!
A part of a structure
May be replaced by something else
Only on condition that
Certain other changes
Are not made in the rest of the structure.

To anyone wishing to pursue the subject further
We recommend the following:
(1) Non-Euclidean Geometry, by F. S. Woods
pubished in Monographs on Modern
Mathematics, edited by J. W. A. Young.
(2) Non-Euclidean Geometry, by R. Bonola.

In these books one may get a splendid
Idea of Non-Euclidean geometry;
Together with a bibliography
For the more serious student.

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VI. The Moral.

Some of the
Most important ideas
Which have been realized
From a study of
Non-Euclidean Geometry
Are:

I. We CANNOT PROVE EVERYTHING,
But must take
Some things for granted. (See p. 28.)

II. Some statements may be true
In one ENVIRONMENT
But not true in another. (See pp. 30 and 31.)

III. Some statements may be true
From one point of view
But not from another! (See p. 3.)

IV. But these things
Need not cause us to despair,—
On the contrary,
They are very fertile ideas,
Since they have led
To the discovery of many geometries
Instead of only one!

V. The human mind
Works very slowly,
For it took many centuries
To realize all this! (See p. 5.)

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Glossary

The meanings of a few important terms Used in this book:

(1) Geodesic: The shortest distance between two points.
Its shape depends upon the surface
On which it is drawn:
Thus, on a Euclidean plane,
A geodesic is a straight line.
On a sphere,
It is an arc of a great circle,
Etc.

(2) Non-Euclidean Geometry: Does not mean
Any Geometry unlike Euclidean,
BUT, SPECIFICALLY,
Either Lobachevskian
Or Riemannian. (See p. 7.)

(3) Parallel lines: Ordinarily we think of
Parallel lines as
Lines that never meet.
In Non-Euclidean geometry
This condition STILL HOLDS
But ANOTHER CONDITION
IS ADDED:
NAMELY;

Two lines, AC and EF, (Fig. 6) are parallel
If they satisfy the two conditions:
(1) They never meet.
(2) Any line, EG,
Drawn within angle FEB
Does meet AC.

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Thus on a pseudo-sphere
Although there is an infinite number
Of lines through E that never meet AC,
Still these are not all called parallel to AC:
Only the limiting one, EF,
Is called parallel to AC. (See p. 9.)

(4) Pseudo-sphere: A surface
Generated by a curve
Called the "tractrix" (Fig. 4.)
It is a surface of constant negative
curvature,
Just as a sphere is a
Surface of constant positive curvature.
(See Bonola: Non-Euclidean Geometry,
p. 132.)

(5) "Straight line": Since a certain path
May seem to have different shapes
Depending on our point of view
(See Fig. 1 and p. 3),
It is preferable
Not to consider its shape at all,
But to think of it merely as
The shortest distance between two
points.
In other words,
To use it in the sense of
"Geodesic" (see above).

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Fig. 12.

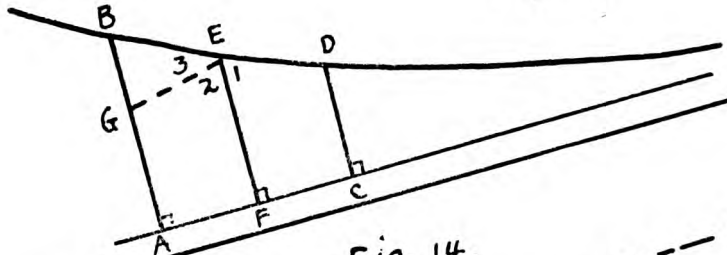


Fig. 14.

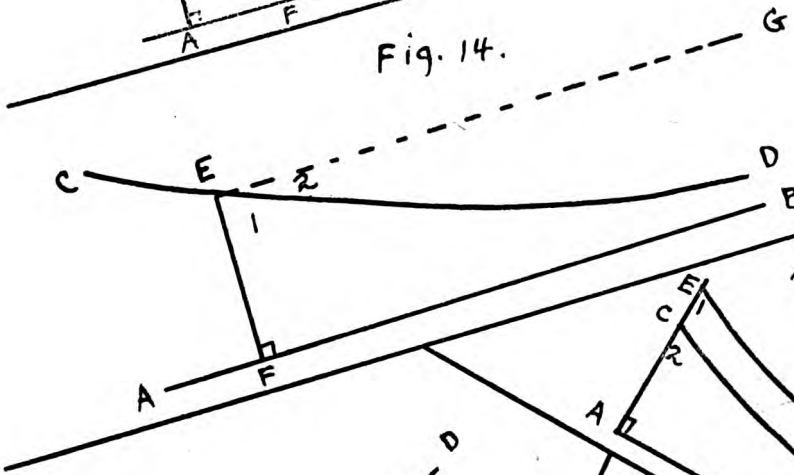
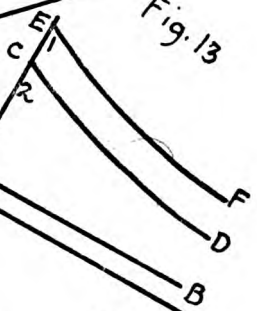


Fig. 13



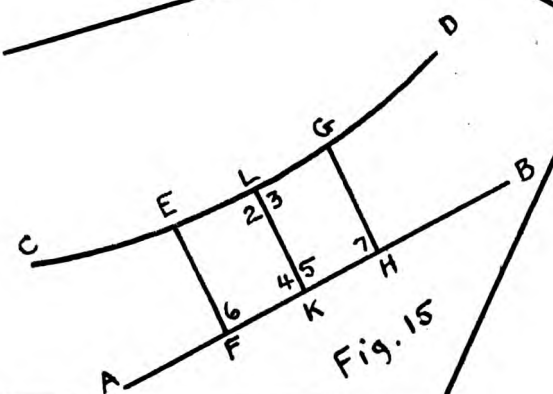


Fig. 15

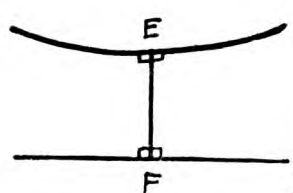


Fig. 17.

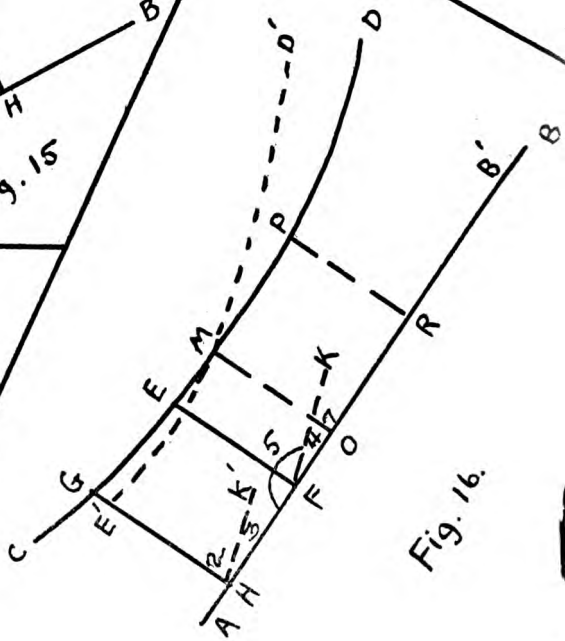


Fig. 16.

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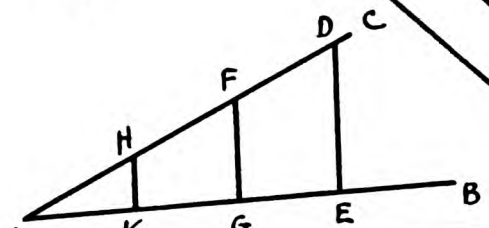
Fig. 18



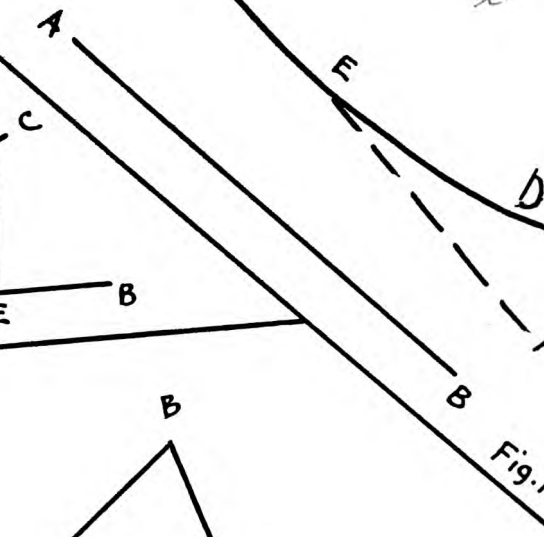


Fig. 20

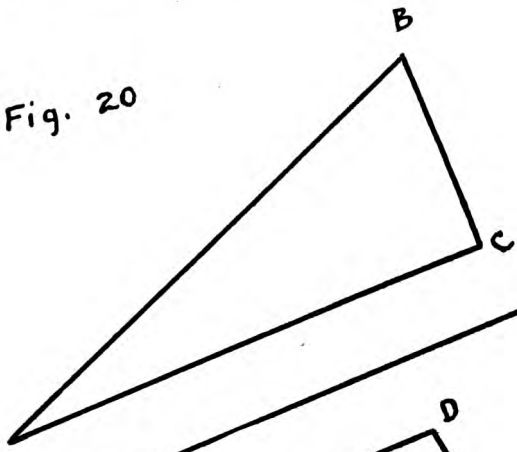
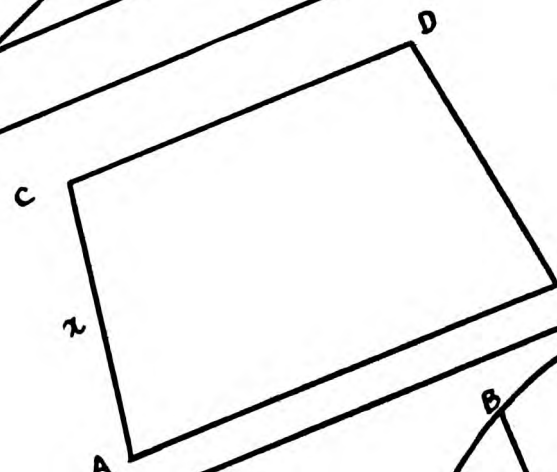
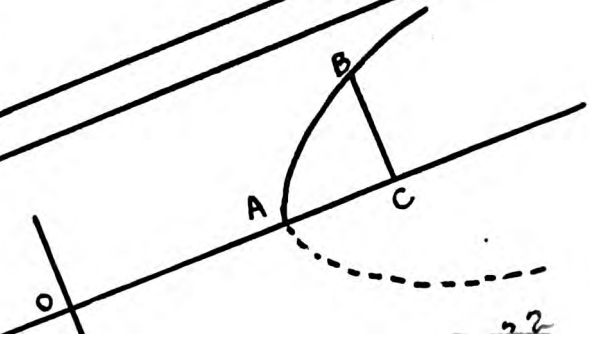


Fig. 21.





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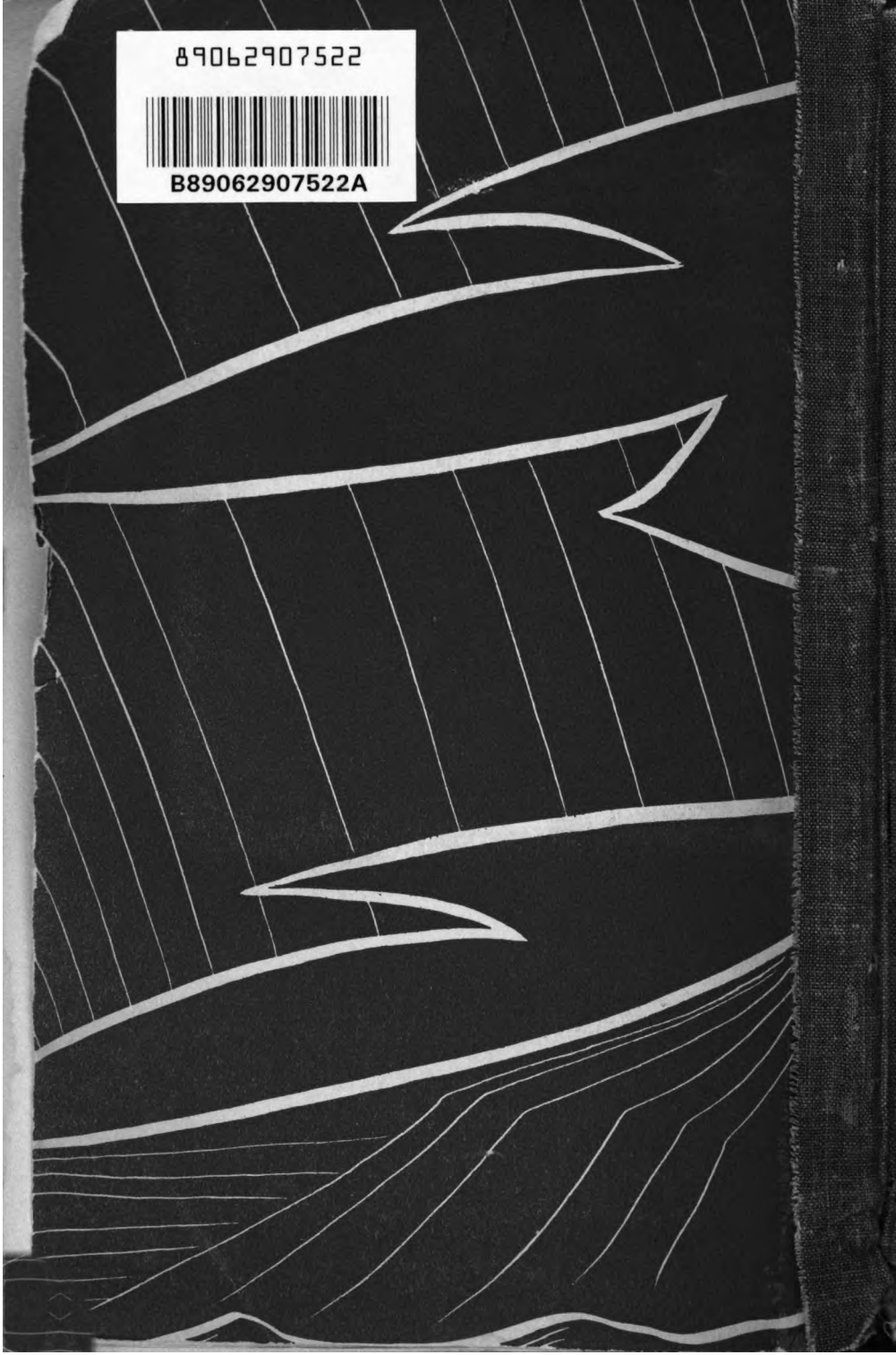
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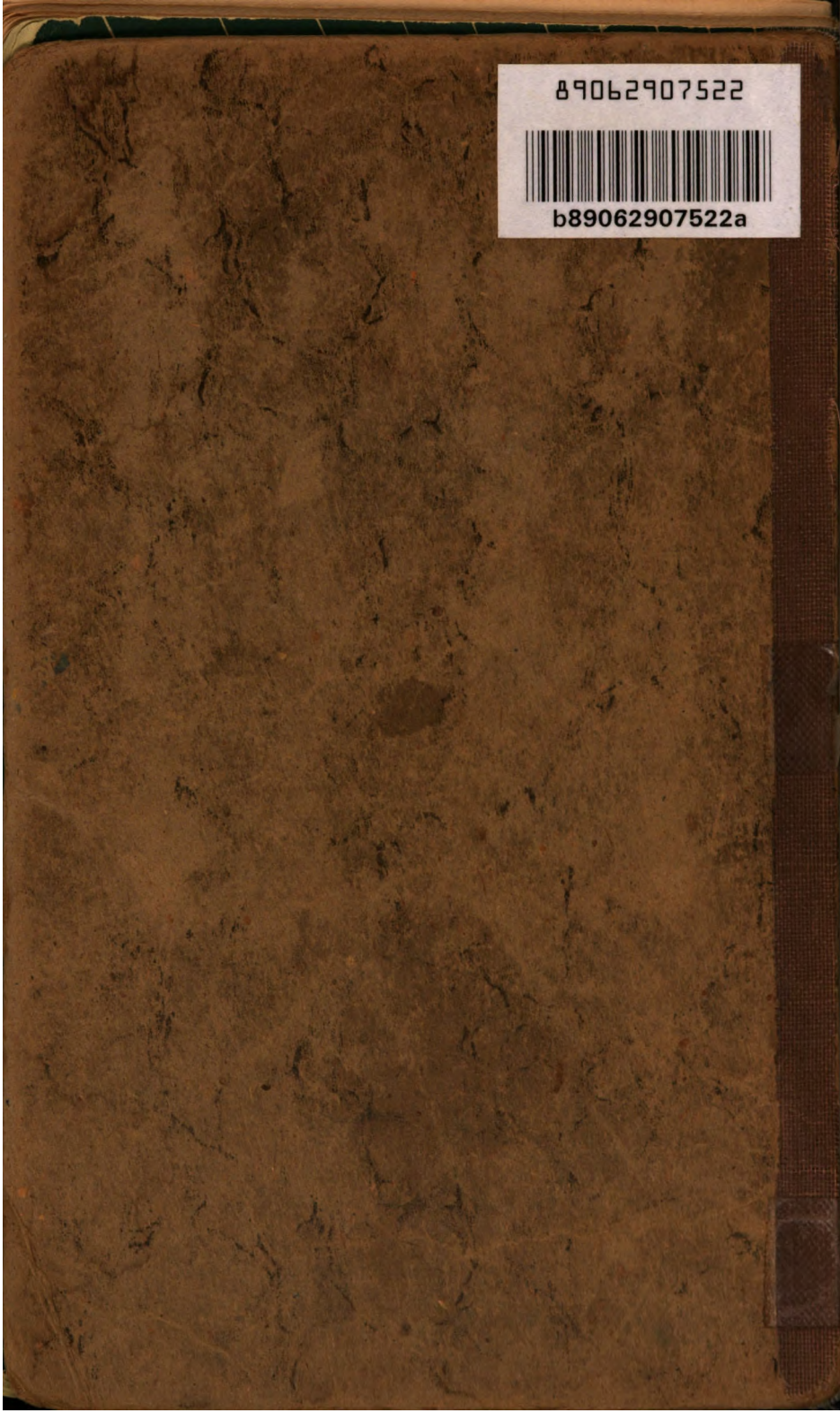
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